Nachweises der ESR [30, 33, 34] wurde im vorliegenden System gezeigt, daß die τ_x-Komponente strahlt [18]. Dieser Befund stimmt mit dem experimentellen Ergebnis $k_x > k_y > k_z$ überein, kann dieses aber nicht quantitativ erklären, denn der *strahlungslose* Anteil der Desaktivierung beträgt etwa 90% [5, 27]. Daraus folgt, daß auch für die strahlungslosen Übergänge, die über hohe Schwingungsanregungen des elektronischen Grundzustandes ^1A_g erfolgen, Auswahlregeln existieren:

$$k_x^{\text{ISC}} > k_y^{\text{ISC}} > k_z^{\text{ISC}} \neq 0 .$$

Diese Auswahlregeln wurden bislang nicht theoretisch behandelt.

Für ein beliebiges System läßt sich der genaue Weg der Be- und Entvölkerung des phosphoreszierenden Triplettzustandes nur vorhersagen, wenn neben der Symmetrie der Elektronenzustände die energetische Lage aller Anregungsniveaus und die Quanten der Molekülschwingungen bekannt sind.

Um die vorliegenden Probleme zu klären, haben wir inzwischen begonnen, weitere Systeme zu untersuchen. Es zeigte sich, daß alle Systeme mit hinreichend „langer" Spin-Gitter-Relaxationszeit Spin-Polarisationseffekte aufwiesen. Das überraschendste Ergebnis war dabei, daß auch bei *Triplett-Excitonen* in reinen Naphthalinkristallen bei etwa 300 °K ein $\Delta m = 1$-Übergang bei spezieller Orientierung emissiv, und einer absorptiv ist [23].

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Heat and Diffusion Fluxes in a Multicomponent Ionized Gas in a Magnetic Field

I. General Expressions

R. S. DEVOTO *

Institut für Plasmaphysik, 8046 Garching bei München

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Ordinary and thermal diffusion as well as heat flux in a dilute, ionized, multicomponent monoatomic gas in a magnetic field are considered with the Chapman-Enskog-Burnett method. It is shown how, with certain modifications, the usual expressions for the properties of an un-ionized monatomic gas may be applied to this case. The expression for the diffusion flux is compared with the momentum equation suggested by Schlüter.

The importance of transport properties in determining the behaviour of an ionized gas has stimulated many theoretical studies of these properties, most often with the Chapman-Enskog-Burnett method [1]. Some expressions for the properties in electric and magnetic fields were given in the first edition of the book by CHAPMAN and COWLING [1], and further work was done by COWLING in 1945 [2], but the first accurate expressions for the electron properties of a fully-ionized gas in a magnetic field were given by LANDSHOFF in 1949 [3]. A number of subsequent studies have confirmed his results [4−8] and extended them to the ion viscosity [5−8], and to the a. c. electrical conductivity [9], and have included the first subdominant (non-logarithmic) term in the expressions [10, 11]. A collection of numerical factors as well as a rule for extending the results for the electron properties in a steady magnetic field to the case of

* National Science Foundation (USA) Postdoctoral Fellow, 1967—1968. Now at Department of Aeronautics and Astronautics, Stanford University, Stanford, California, USA.

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combined magnetic and alternating electric fields have been given [12].

The inclusion of neutral species into accurate expressions for the transport properties has been effected only recently. COWLING [2] included neutral species in the first approximation [13] to the electrical conductivity in a magnetic field. This approximation is, however, highly inaccurate for the strongly ionized gas ($\gtrsim 1\%$ ionization), and predicts much too small a coefficient at low ionization for certain types of electron-neutral cross sections [14, 15]. Further work included development of expressions for electron coefficients in a three-component plasma without [14] and with a magnetic field [16]. Explicit expressions are now available for up to the sixth approximation to all electron coefficients in multicomponent mixtures in a magnetic field in which the electron-neutral cross sections have an arbitrary dependance on relative energy [17, 18]. The behaviour of the ion and neutral species in a multicomponent plasma without B-field have also been considered [17, 19, 20].

The purpose of the present work is to extend the expressions for the transport coefficients of a mixture of an arbitrary number of species [19, 21], both charged and uncharged, to the case where a magnetic field is present. These expressions turn out to be only slightly different from those previously derived for this gas in the absence of a magnetic field. As with the electron properties in a magnetic field [1, 2, 16–18, 22], it is necessary only to introduce complex terms containing the various cyclotron frequencies, and then to extract the real and complex parts of the coefficients. The former is the transverse component, perpendicular to the magnetic field $\boldsymbol{B}$ and the driving force under consideration ($\boldsymbol{E}$, ∇p_i, etc.) and the latter is the so-called Hall component-perpendicular to both $\boldsymbol{B}$ and to the forcing term. The coefficients parallel to $\boldsymbol{B}$ are the limit of the complex expressions when $|\boldsymbol{B}| \to 0$.

In the expressions all species are considered on an equal basis, i. e. the simplifications possible because of the small electron mass have not been introduced. As a result they are also applicable to gases composed of positive and negative ions. More often, the gas under consideration will contain the very much lighter electrons in a concentration equal to or greater than that of the ions. In this case the electron properties will be affected by a much smaller magnetic field than that necessary to influence the ions. Since it is possible to develop much simplified expressions for the electron properties in multicomponent gases [16–18, 22], it is possible in this case to compute the influence of the magnetic field without using the more complicated expressions developed here. In an increasing number of experiments, e. g. magnetically confined electric arcs [23, 24], the magnetic field is so large that the electron coefficient is reduced below that of the ions, and the latter itself is reduced by the magnetic field. In this case, the expressions presented here are useful in predicting the coefficients in regions of partial ionization, or where ions of several sorts are present. SCHLÜTER [25] and, slightly later, JOHNSON [26] have suggested that a type of momentum equation for each species be used to predict the motion of individual species in an ionized gas. Since these expressions have found wide application in understanding the behaviour of electric arcs [23, 27, 28], it is worthwhile to see if they can be justified from kinetic theory. A comparison of the results of kinetic theory with the expressions of SCHLÜTER is also given here.

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Solution of the Linearized Boltzmann Equation

The solution of the Boltzmann equation follows along the same lines as given for the multicomponent gas without magnetic field [21] and for certain properties of the binary ionized gas in a magnetic field [1]. Since the theoretical basis of the derivation is discussed at length in these two references, it will not be repeated here. Only enough of the steps will be given to establish connection with the derivation of the properties of the un-ionized multicomponent gas.

The properties are all given in terms of integrals over the perturbation φ_i to the equilibrium Maxwellian distribution $f_i^{(0)}$, which is found from the ν coupled equations (c. g. s. units) [1, 21]

$$f_i^{(0)}\left[\frac{n}{n_i}\,\boldsymbol{V}_i\cdot\boldsymbol{d}_i+2\,\mathbf{b}_i:\nabla\boldsymbol{v}_0-(\tfrac{5}{2}-W_i^2)\,\boldsymbol{V}_i\cdot\nabla\ln T\right]$$
$$=-f_i^{(0)}\left[\frac{m_i}{\varrho_c\,k\,T}\,\boldsymbol{V}_i\cdot(\boldsymbol{j}\times\boldsymbol{B})+\frac{e_i}{m_i\,c}\frac{\partial\varphi_i}{\partial\boldsymbol{V}_i}\cdot(\boldsymbol{V}_i\times\boldsymbol{B})\right]$$
$$-n_i\sum_j n_j\,I_{ij}(\varphi_i+\varphi_j) \qquad (1)$$

with

$$n_i\,n_j\,I_{ij}(\varphi_i)=\iint f_i^{(0)}\,f_j^{(0)}\,(\varphi_i-\varphi_i')\,g\,\sigma\,\mathrm{d}\Omega\,\mathrm{d}\boldsymbol{v}_j \qquad (2)$$

and

$$\frac{p\,\boldsymbol{d}_i}{n_i\,m_i}=\frac{\nabla p_i}{n_i\,m_i}-\frac{\nabla p}{\varrho}-\left(\frac{e_i}{m_i}-\frac{\varrho_c}{\varrho}\right)\left(\boldsymbol{E}+\frac{\boldsymbol{v}_0\times\boldsymbol{B}}{c}\right)$$
$$-\frac{1}{\varrho}\left[\frac{\varrho}{m_i}\boldsymbol{X}_i-\sum_k n_k\,\boldsymbol{X}_k\right]. \qquad (3)$$

$\boldsymbol{j}$ is the electric current, e_i is the charge on the i^{th} species, $\boldsymbol{B}$ is the magnetic field, $\varrho_c=\sum_k e_k\,n_k$ is the charge density (negligible in most gases), and $\boldsymbol{X}_i$ represents all external forces not electromagnetic in nature. Other symbols are those used in Ref. [21]. When not otherwise indicated, the sums run over all species i. e. from 1 to ν. For later purposes it is worth noting that the thermal velocity $\boldsymbol{V}_i$ is related to the molecular velocity $\boldsymbol{v}_i$ and the mass average velocity $\boldsymbol{v}_0$ by $\boldsymbol{V}_i=\boldsymbol{v}_i-\boldsymbol{v}_0$.

Magnetic interactions as well as collisions are considered on an equal basis in Eq. (1); situations in which the magnetic effects are dominant as well as those in which collisions are dominant can be treated with the same theory. Since the masses and charges of the several components in, for example, a partially ionized plasma can be considerably different, it is clear that collisions may govern the transport behaviour of certain species, while finite gyroradius effects govern that of others.

It should be noted that the Boltzmann form of the collision integral used here Eq. (2) is, strictly speaking, valid neither for the case of charged-particle interactions, nor in very high magnetic fields, when the field affects the collisions. In the former case, it is well-known that the Boltzmann term does correctly describe charged-particle interactions to dominant order, i. e. large Coulomb logarithm, when either the Coulomb potential with cut-off at the Debye length or the Debye-shielded Coulomb potential is used. The incorporation of results from accurate collision terms [10, 11, 29] into expressions derived with the Boltzmann form is straight forward [30]. It should be possible to do the same with collision integrals including the effect of a finite Larmour radius [31, 32]. Under a very wide range of conditions such effects may be ignored [33].

The term involving $\mathbf{b}_i$ in Eq. (1) contributes only to the stress tensor and, for the purposes of this paper, can be ignored. The solution to the remainder of Eq. (1) can be written in the form

$$\varphi_i=-\boldsymbol{A}_i^{\|}\cdot\nabla^{\|}\ln T-\boldsymbol{A}_i^{\perp}\cdot\nabla^{\perp}\ln T-\boldsymbol{A}_i^{\mathrm{H}}\cdot\hat{\boldsymbol{B}}\times\nabla^{\perp}\ln T$$
$$+n\sum_j\left[\boldsymbol{C}_i^{j\|}\cdot\boldsymbol{d}_j^{\|}+\boldsymbol{C}_i^{j\perp}\cdot\boldsymbol{d}_j^{\perp}+\boldsymbol{C}_i^{j\mathrm{H}}\cdot\hat{\boldsymbol{B}}\times\boldsymbol{d}_j^{\perp}\right] \qquad (4)$$

with $\boldsymbol{A}_i^{\|}=A_i^{\|}\,\boldsymbol{W}_i$, $\boldsymbol{A}_i^{\perp}=A_i^{\perp}\,\boldsymbol{W}_i$, etc. The scalars $A_i^{\|}$, $A_i^{\perp}$, etc. are functions of W_i^2 and B^2, and of the thermodynamic state of the mixture, which in the Chapman-Enskog theory is assumed given by the fluid dynamical equations. $\hat{\boldsymbol{B}}$ is the unit vector $\boldsymbol{B}/B$ and the superscripts $\|$, $\perp$ and H denote the components parallel to $\boldsymbol{B}$ (e. g. $\hat{\boldsymbol{B}}\cdot\nabla T$), perpendicular to $\boldsymbol{B}$ but parallel to the driving force e. g.

$$\nabla T-\hat{\boldsymbol{B}}(\hat{\boldsymbol{B}}\,\nabla T)=-\hat{\boldsymbol{B}}\times(\hat{\boldsymbol{B}}\times\nabla T),$$

and in the Hall direction (e. g. $\hat{\boldsymbol{B}}\times\nabla^{\perp}T$). The form chosen here for φ_i may be shown via vector manipulation to be the same as that given elsewhere [1, 5]. With this form for φ_i the electric current

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can be written as

$$\boldsymbol{j} = \sum_i e_i n_i \langle \boldsymbol{V}_i \rangle$$
$$= -M^{\|} \nabla^{\|} \ln T - M^{\perp} \nabla^{\perp} \ln T - M^{\mathrm{H}} \hat{\boldsymbol{B}} \times \nabla^{\perp} \ln T + n \sum_j (L^{j\|} \boldsymbol{d}_j^{\|} + L^{j\perp} \boldsymbol{d}_j^{\perp} + L^{j\mathrm{H}} \hat{\boldsymbol{B}} \times \boldsymbol{d}_j^{\perp}), \tag{5}$$

where the coefficients are given by

$$M^{\|} = \sum_i \tfrac{1}{3} e_i (m_i/2\,k\,T)^{1/2} \int f_i^{(0)} A_i^{\|} V_i^2 \, \mathrm{d}\boldsymbol{V}_i , \tag{6}$$

$$L^{j\|} = \sum_i \tfrac{1}{3} e_i (m_i/2\,k\,T)^{1/2} \int f_i^{(0)} C_i^{j\|} V_i^2 \, \mathrm{d}\boldsymbol{V}_i , \tag{7}$$

and similar relations for the other directions.

The solution to Eq. (1) is linear in the various gradients, which are determined by the equations of magnetohydrodynamics. The gradients may be considered as independent parameters in Eq. (1), and the coefficient of each must vanish. We need consider only the solutions for $C_i^{j\|}$, $C_i^{j\perp}$ and $C_i^{j\mathrm{H}}$, since the solutions for $A_i^{\|}$, $A_i^{\perp}$ and A_i^{H} proceed via almost identical steps. From Eqs. (4) – (7), neglecting terms involving $\nabla^{\|} \ln T$ and $\nabla^{\perp} \ln T$, we obtain

$$\frac{\partial \varphi_i}{\partial \boldsymbol{V}_i} \cdot \boldsymbol{V}_i \times \boldsymbol{B} = -n\,B \sum_j (\boldsymbol{C}_i^{j\perp} \cdot \boldsymbol{d}_j^{\perp} - \boldsymbol{C}_i^{j\mathrm{H}} \cdot \hat{\boldsymbol{B}} \times \boldsymbol{d}_j^{\perp}) \tag{8}$$

and

$$\boldsymbol{j} \times \boldsymbol{B} = n\,B \sum_j (L^{j\mathrm{H}} \boldsymbol{d}_j^{\perp} - L^{j\perp} \hat{\boldsymbol{B}} \times \boldsymbol{d}_j^{\perp}) . \tag{9}$$

As could be anticipated, we see that $C_i^{j\|}$ does not occur in either Eq. (8) or (9). The equation for this component therefore does not contain the magnetic field, and is identical to that obtained in the absence of this field. The solution for $C_i^{j\|}$ and the corresponding transport coefficients parallel to the magnetic field are just those already given for the multicomponent gas [21, 34]. Equivalent remarks apply to $A_i^{\|}$.

Upon substituting Eqs. (8) and (9) into Eq. (1), making use of the identity $\sum_j \boldsymbol{d}_j = 0$ [21], and equating coefficients of $\boldsymbol{d}_h^{\perp}$ and $\hat{\boldsymbol{B}} \times \boldsymbol{d}_h^{\perp}$, we obtain

$$\frac{1}{n_i} f_i^{(0)} (\delta_{ih} - \delta_{ik}) \, \boldsymbol{V}_i = -f_i^{(0)} \left[\frac{B}{\varrho_{\mathrm{c}}} \left(\frac{2\,m_i}{k\,T} \right)^{1/2} (L^{h\mathrm{H}} - L^{k\mathrm{H}}) - \omega_i (C_i^{h\mathrm{H}} - C_i^{k\mathrm{H}}) \right] \boldsymbol{W}_i - n_i \sum_j n_j I_{ij} (\boldsymbol{C}_i^{h\perp} + \boldsymbol{C}_j^{h\perp} - \boldsymbol{C}_i^{k\perp} - \boldsymbol{C}_j^{k\perp}), \tag{10}$$

$$0 = f_i^{(0)} \left[\frac{B}{\varrho_{\mathrm{c}}} \left(\frac{2\,m_i}{k\,T} \right)^{1/2} (L^{h\perp} - L^{k\perp}) - \omega_i (C_i^{h\perp} - C_i^{k\perp}) \right] \boldsymbol{W}_i - n_i \sum_j n_j I_{ij} (\boldsymbol{C}_i^{h\mathrm{H}} + \boldsymbol{C}_j^{h\mathrm{H}} - \boldsymbol{C}_i^{k\mathrm{H}} - \boldsymbol{C}_j^{k\mathrm{H}}), \tag{11}$$

with $\omega_i = e_i B / m_i c$, the cyclotron frequency of species i. In the limit of vanishing B-field, $C_i^{h\mathrm{H}}$, and therefore $L^{h\mathrm{H}}$ vanish [35]. We see that, in this limit, Eq. (10) becomes identical with that for $C_i^{h\|} - C_i^{k\|}$ so that

$$\lim_{B \to 0} (C_i^{h\perp} - C_i^{k\perp}) = C_i^{h\|} - C_i^{k\|} . \tag{12}$$

All parallel transport coefficients may therefore be computed from the perpendicular coefficients by taking the limit of vanishing magnetic field.

Equations (10) and (11) are coupled and are most easily solved by introducing the new complex variable [1, 34]

$$\boldsymbol{C}_i^{\mathrm{h}} \equiv C_i^{\mathrm{h}} \boldsymbol{W}_i = (C_i^{h\perp} + i\,C_i^{h\mathrm{H}}) \, \boldsymbol{W}_i \tag{13}$$

and the complex coefficient

$$L^h = L^{h\perp} + i\,L^{h\mathrm{H}} . \tag{14}$$

Multiplying Eq. (11) by $i = \sqrt{-1}$ and adding the result to Eq. (11) we obtain

$$\frac{1}{n_i} f_i^{(0)} (\delta_{ih} - \delta_{ik}) \, \boldsymbol{V}_i = i\,f_i^{(0)} \left[\frac{B}{\varrho_{\mathrm{c}}} \left(\frac{2\,m_i}{k\,T} \right)^{1/2} (L^h - L^k) - \omega_i (C_i^h - C_i^k) \right] \boldsymbol{W}_i - n_i \sum_j n_j I_{ij} (\boldsymbol{C}_i^h + \boldsymbol{C}_j^h - \boldsymbol{C}_i^k - \boldsymbol{C}_j^k), \tag{15}$$

which may be solved in the same way as in the absence of a magnetic field. We substitue in Eq. (15) the finite expansion

$$C_i^h - C_i^k = \sum_{p=0}^{\xi-1} c_{ip}^{hk} S_{3/2}^{(p)} (W_i^2) , \tag{16}$$

where $S_{3/2}^{(p)} (W_i^2)$ are Sonine polynomials [1, 3, 21] and the coefficients c_{ip}^{hk} are, of course, complex. Now form the inner product of $(2\,\pi\,m_i/k\,T)^{1/2} S_{3/2}^{(m)} (W_i^2) \, \boldsymbol{W}_i$ with the resultant equation, integrate over $\mathrm{d}\boldsymbol{V}_i$ and rearrange to get

$$\sum_j \sum_{p=0}^{\xi-1} q_{ij}^{mp} c_{jp}^{hk} = 3\,\pi^{1/2} \delta_{m0} (\delta_{ik} - \delta_{ih}) \qquad (m = 0, 1, \ldots, \xi - 1) \tag{17}$$

[34] All coefficients in the absence of a magnetic field will be denoted by the superscript $\|$. Symbols without this superscript in the present paper always denote complex combinations of perpendicular ($\perp$) and Hall (H) coefficients, e. g. $C_i^j \equiv C_i^{j\perp} + i\,C_i^{j\mathrm{H}}$.

[35] Comparing the form for Φ_i with that of Ref. [1] we see that $C_i^{j\mathrm{H}}$ must be B times a scalar function of W_i^2 and B^2. If $C_i^{j\mathrm{H}}$ were proportional to B^{-2}, then the Hall electrical conductivity, for example, would become infinite at $B = 0$, which is physically impossible.

with

$$\underline{q}_{ij}^{mp} = q_{ij}^{mp} + i\,\omega_j\,n_j \left(\frac{2\,\pi\,m_j}{k\,T}\right)^{1/2} \frac{2\,(p+\frac{3}{2})\,!}{\pi^{1/2}\,p!}\,\delta_{mp} \cdot \cdot \left(\delta_{ij} - \frac{n_i\,m_i}{\varrho}\,\delta_{m0}\right). \quad (18)$$

The elements q_{ij}^{mp} are related to Q_{ij}^{mp} elements introduced in HIRSCHFELDER et al. [21] by

$$q_{ij}^{mp} = (2\,\pi\,m_i/k\,T)^{1/2}\,Q_{ij}^{mp} \quad (19)$$

and given in terms of the bracket integrals [36] by

$$q_{ii}^{mp} = n_i \left(\frac{2\,\pi\,m_i}{k\,T}\right)^{1/2} \{ n_i [S_{3/2}^{(m)}(W_i^2)\,\mathbf{W}_i;\, S_{3/2}^{(p)}(W_i^2)\,\mathbf{W}_i]_i + \sum_{j \neq i} n_j [S_{3/2}^{(m)}(W_i^2)\,\mathbf{W}_i;\, S_{3/2}^{(p)}(W_i^2)\,\mathbf{W}_i]_{ij} \} \quad (20)$$

and

$$q_{ij}^{mp} = n_i\,n_j \left(\frac{2\,\pi\,m_i}{k\,T}\right)^{1/2} [S_{3/2}^{(m)}(W_i^2)\,\mathbf{W}_i;\, S_{3/2}^{(p)} (W_j^2)\,\mathbf{W}_j]_{ij}\,(i \neq j). \quad (21)$$

The only effect of the magnetic field occurs in the imaginary part of Eq. (18).

As in the case of the multicomponent gas without a magnetic field, the Eqs. (18) with $m=0$ are not linearly independent, but contain one redundant equation [37]. To prove this we need show only that the coefficients of one of the equations can be written as a linear combination of the coefficients of the others. It can be shown that

$$\underline{q}_{ki}^{op} = -\sum_{k \neq i} \underline{q}_{kj}^{op}. \quad (22)$$

The proof of this equality follows from two properties of the bracket integrals,

$$[(0)_i;\,(p)_i]_i = 0$$

and $$m_i^{1/2}[(0)_i;\,(p)_i]_{ij} = -m_j^{1/2}[(0)_j;\,(p)_i]_{ji}$$

where obvious symbolic notation has been used for the bracket integrals of Eqs. (20) and (21). These properties can be deduced from relations given in Chap. 9 of Ref. [1]. The redundancy of Eqs. (17) for $m=5$ can be removed with the aid of the auxiliary condition

$$\varrho\,\boldsymbol{v}_0 = \sum_j m_j \int \boldsymbol{v}_j\,f_j^{(0)}\,(1+\varphi_j)\,\mathrm{d}\boldsymbol{v}_j\,, \quad (23)$$

where $\boldsymbol{v}_0$ is the mass average velocity of the gas. Upon insertion of the form for φ_j, the condition on the C_i^j becomes

$$\begin{aligned}\sum_j m_j \left(\frac{m_j}{2\,k\,T}\right)^{1/2} \sum_{h \neq k} [\boldsymbol{d}_h^{\parallel} \int f_j^{(0)}\,(C_j^{h\parallel} - C_j^{k\parallel})\,V_j^2\,\mathrm{d}\mathbf{V}_j \\ + \boldsymbol{d}_h^{\perp} \int f_j^{(0)}\,(C_j^{h\perp} - C_j^{k\perp})\,V_j^2\,\mathrm{d}\mathbf{V}_j \quad (24) \\ + (\hat{\boldsymbol{B}} \times \boldsymbol{d}_h^{\perp}) \int f_j^{(0)}\,(C_j^{h\mathrm{H}} - C_j^{k\mathrm{H}})\,V_j^2\,\mathrm{d}\mathbf{V}_j] = 0\,.\end{aligned}$$

This condition must hold for any orientation of $\boldsymbol{d}_j$ relative to $\boldsymbol{B}$, so the sum over j of each term must separately vanish. The first sum is that used in the ordinary gas mixture [21]. The second and third can be combined into the complex condition.

$$\sum_j m_j \left(\frac{m_j}{2\,k\,T}\right)^{1/2} \int f_j^{(0)}\,(C_j^h - C_j^k)\,V_j^2\,\mathrm{d}\mathbf{V}_j = 0\,. \quad (25)$$

Substituting the expansion of Eq. (16) we obtain

$$\sum_j n_j\,m_j^{1/2}\,c_{j0}^{hk} = 0\,. \quad (26)$$

This equation is then multiplied by $\underline{q}_{ii}^{00}$, divided by $n_i\,m_i^{1/2}$ and subtracted from Eq. (17) to yield the independent equations

$$\sum_j \sum_{p=0}^{\xi-1} \tilde{\underline{q}}_{ij}^{mp}\,c_{jp}^{hk} = 3\,\pi^{1/2}\,(\delta_{ik} - \delta_{ih}) \quad (m = 0, 1, \ldots, \xi-1) \quad (27)$$

with

$$\tilde{\underline{q}}_{ij}^{mp} = \underline{q}_{ij}^{mp} - \frac{n_j}{n_i}\left(\frac{m_j}{m_i}\right)^{1/2} \underline{q}_{ii}^{mp}\,\delta_{m0}\,\delta_{p0}\,. \quad (28)$$

Some particular forms of Eq. (28) are:

$$\underline{q}_{ii}^{00} = 0\,, \quad (28\text{ a})$$

$$\begin{aligned}\tilde{\underline{q}}_{ij}^{00} &= -8\,n_i\,n_j \left(\frac{m_i}{m_i+m_j}\right)^{1/2} \overline{Q}_{ij}^{(1,1)} \\ &\quad - 8\left(\frac{m_j}{m_i}\right)^{1/2} n_j \sum_{k \neq i} n_k \left(\frac{m_k}{m_i+m_k}\right)^{1/2} \overline{Q}_{ik}^{(1,1)} \quad (28\text{ b}) \\ &\quad - i\,\frac{3\,n_j}{2}\left(\frac{2\,\pi\,m_j}{k\,T}\right)^{1/2} \left[\frac{n_i\,m_i}{\varrho}\,(\omega_j - \omega_i) + \omega_i\right],\end{aligned}$$

$$\tilde{\underline{q}}_{ii}^{mm} = q_{ii}^{mm} + i\,\omega_i\,n_i \left(\frac{2\,\pi\,m_i}{k\,T}\right)^{1/2} \frac{2\,(m+\frac{3}{2})\,!}{\pi^{1/2}\,m!} \quad (m>0)\,, \quad (28\text{ c})$$

$$\tilde{\underline{q}}_{ij}^{mm} = q_{ij}^{mm}\,(m>0,\; i \neq j)\,, \quad (28\text{ d})$$

$$\tilde{\underline{q}}_{ij}^{mp} = q_{ij}^{mp}\,(m \neq p,\; m>0)\,. \quad (28\text{ e})$$

We see that the magnetic field affects all elements for $m=p=0$, but occurs otherwise only in diagonal elements. Thus only minor additions to the usual

[36] Ref. [1], p. 85–88 and Chap. 9.

[37] Mentioned in Ref. [21], p. 477, but the proof was not given.

determinant elements are necessary to include the effect of a magentic field. Expressions for the q elements in terms of average cross section

$$\widetilde{Q}^{(l,s)} = \pi \sigma^2 \Omega^{(l,s)*}$$

have been given elsewhere [19] for $m, p \leqq 3$. Thus, up to the fourth approximation to the coefficients can be computed.

As already mentioned, the solution for $A_i^{\perp}$ and A_i^{H} will not be considered in detail. Suffice it to say that the coefficients a_{im} in the expansion

$$A_i \equiv A_i^{\perp} + i\, A_i^{\mathrm{H}} = \sum_{m=0}^{\xi-1} a_{im}\, S_{3/2}^{(m)}(W_i^2) \tag{29}$$

are found from

$$\sum_j \sum_{p=0}^{\xi-1} q_{ij}^{mp}\, a_{ip} = -\frac{15\,\pi^{1/2}}{2}\, n_i\, \delta_{m1}\,. \tag{30}$$

Transport Coefficients

In view of the similarity of the results of the previous section to those for the usual multitcomponent mixture, it is not surprising that the usual expressions for the transport coefficients carry over into this case, with the simple addition of complex terms.

Diffusion Velocity

The diffusion velocity of the i^{th} species is given by

$$\begin{aligned}\langle \mathbf{V}_i \rangle &\equiv \frac{1}{n_i} \int \mathbf{V}_i\, f_i^{(0)}\, \varphi_i\, \mathrm{d}\mathbf{V}_i \\ &= \frac{n^2}{n_i\, \varrho} \sum_j m_j [D_{ij}^{\parallel}\, \mathbf{d}_j^{\parallel} + D_{ij}^{\perp}\, \mathbf{d}_j^{\perp} + D_{ij}^{\mathrm{H}} (\hat{\mathbf{B}} \times \mathbf{d}_j^{\perp})] \\ &\quad - \frac{1}{n_i\, m_i\, T} (D_i^{\mathrm{T}\parallel}\, \nabla^{\parallel} T + D_i^{\mathrm{T}\perp}\, \nabla^{\perp} T \\ &\quad + D_i^{\mathrm{TH}}\, \hat{\mathbf{B}} \times \nabla^{\perp} T),\end{aligned} \tag{31}$$

with the perpendicular and Hall components of the diffusion coefficients found from

$$D_{ij} \equiv D_{ij}^{\perp} + i\, D_{ij}^{\mathrm{H}} = \frac{\varrho\, n_i}{2\, n\, m_j} \left(\frac{2\,k\,T}{m_i} \right)^{1/2} c_{io}^{ji} \tag{32}$$

and

$$D_i^{\mathrm{T}} \equiv D_i^{\mathrm{T}\perp} + i\, D_i^{\mathrm{TH}} = \frac{n_i\, m_i}{2} \left(\frac{2\,k\,T}{m_i} \right)^{1/2} a_{i0}\,. \tag{33}$$

We see that both complex coefficients are given by the usual multicomponent expressions [38] with the replacement of certain of the real determinant elements by the complex elements of Eq. (27). Thus the computation of properties in a magnetic field may be carried out with little more effort than that necessary for the un-ionized multicomponent gas. The task becomes quite simple on computers where complex arithmetic is part of the programming language. The parallel diffusion coefficients are then simply found from the perpendicular components with vanishing magnetic field.

An alternative formulation of Eq. (31) including the binary diffusion coefficients has been obtained by HIRSCHFELDER et al. [39]. This relation applies here to the parallel diffusion velocities and reads,

$$\begin{aligned}&\sum_{j \neq i} \frac{n_i\, n_j}{n^2\, \mathcal{D}_{ij}} (\langle \mathbf{V}_j^{\parallel} \rangle - \langle \mathbf{V}_i^{\parallel} \rangle) \\ &= \mathbf{d}_i^{\parallel} - \sum_{j \neq i} \frac{n_i\, n_j}{n^2\, T\, \mathcal{D}_{ij}} \left(\frac{D_j^{\mathrm{T}\parallel}}{n_j\, m_j} - \frac{D_i^{\mathrm{T}\parallel}}{n_i\, m_i} \right) \nabla^{\parallel} T\end{aligned} \tag{34}$$

where the first approximation to the binary diffusion coefficient is given by

$$\mathcal{D}_{ij} = \frac{3}{16\, n} \left(\frac{m_i + m_j}{m_i\, m_j} \right)^{1/2} \frac{(2\,\pi\,k\,T)^{1/2}}{\bar{Q}_{ij}^{(1,1)}}\,. \tag{35}$$

If thermal diffusion is neglected, as it may be in many experiments, then Eq. (35) is much simpler than the corresponding form of Eq. (31), since the expressions for D_{ij} need not be evaluated. A similar expression may also be derived for the gas in a magentic field. We define a complex diffusion velocity

$$\langle \mathbf{V}_i^* \rangle = \langle \mathbf{V}_i^{\perp} \rangle - i\, \hat{\mathbf{B}} \times \langle \mathbf{V}_i^{\perp} \rangle \tag{36}$$

and complex driving forces

$$\mathbf{d}_j^* = \mathbf{d}_j^{\perp} - i\, \hat{\mathbf{B}} \times \mathbf{d}_j^{\perp}, \tag{37}$$

$$\nabla^* T = \nabla^{\perp} T - i\, \hat{\mathbf{B}} \times \nabla^{\perp} T \tag{38}$$

and find from Eq. (31) that

$$\sum_{j \neq i} \frac{n_i\, n_j}{\mathcal{D}_{ij}} (\langle \mathbf{V}_i^* \rangle - \langle \mathbf{V}_j^* \rangle) = \frac{n^2}{\varrho} \sum_{\substack{j,k \\ j \neq i}} \frac{m_k}{\mathcal{D}_{ij}} (n_j\, D_{ik} - n_i\, D_{jk})\, \mathbf{d}_k^* - \sum_{j \neq i} \frac{n_i\, n_j}{T\, \mathcal{D}_{ij}} \left(\frac{D_i^{\mathrm{T}}}{n_i\, m_i} - \frac{D_j^{\mathrm{T}}}{n_j\, m_j} \right) \nabla^* T, \tag{39}$$

[38] See Eqs. (8) and (9) of Ref. [19], or the corresponding equations in Ref. [21].

[39] Ref. [21], p. 487.

where the diffusion coefficients D_{ij} and D_i^{T} are, of course, complex. In the first approximation $(m=0)$ Eq. (27) can be rewritten in the form

$$\sum_{j \neq i} \left\{ \frac{n_i}{\mathcal{D}_{ij}} + \frac{m_j}{m_i} \sum_{k \neq i} \frac{n_k}{\mathcal{D}_{ik}} + i \frac{n\, m_j}{k\,T} \left[\frac{n_i\, m_i}{\varrho} (\omega_j - \omega_i) + \omega_i \right] \right\} [m_h D_{jh}(1) - m_k D_{jk}(1)] = \varrho (\delta_{ih} - \delta_{ik}), \tag{40}$$

where the symbol (1) indicates the first approximation is involved. From Eqs. (26) and (32) we obtain

$$\sum_j m_j (m_h D_{jh} - m_k D_{jk}) = 0 \tag{41}$$

which can be used with $\sum_k d_k^* = 0$ and Eq. (40) to obtain

$$\sum_{\substack{j,k \\ j \neq i}} \frac{m_k}{\mathcal{D}_{ij}} [n_j D_{ik}(1) - n_i D_{jk}(1)]\, \mathbf{d}_k^* = -\varrho\, \mathbf{d}_i^* + i \frac{n}{k\,T} \left[\frac{n_i\, m_i}{\varrho} \sum_{j,k} \frac{e_j B}{c} m_k D_{jk}\, \mathbf{d}_k^* - \frac{e_i B}{c} \sum_k m_k D_{ik}\, \mathbf{d}_k^* \right] \tag{42}$$

Equation (42) may now be used in Eq. (39) to eliminate the first term on the right hand side. The resultant expression still contains terms involving D_{jk}, which may be eliminated in favour of $\boldsymbol{j} \times \boldsymbol{B}$ and $e_i n_i \langle \boldsymbol{V}_i \rangle \times \boldsymbol{B}$ with Eq. (31). The real part of the final expression reads

$$\begin{aligned}
&\sum_{j \neq i} \frac{n_i n_j}{n^2 \mathcal{D}_{ij}} (\langle \mathbf{V}_j^{\perp} \rangle - \langle \mathbf{V}_i^{\perp} \rangle) = \mathbf{d}_i^{\perp} + \frac{1}{c\,n\,k\,T} \left[\frac{n_i\, m_i}{\varrho} (\boldsymbol{j} \times \boldsymbol{B}) - e_i n_i (\langle \boldsymbol{V}_i \rangle \times \boldsymbol{B}) \right] \\
&+ \frac{1}{n\,k\,T^2} \left\{ \frac{n_i\, m_i}{\varrho} \sum_j \omega_j [D_j^{\mathrm{T}\perp} (\nabla^{\perp} T \times \boldsymbol{B}) + D_j^{\mathrm{TH}} \nabla^{\perp} T] - \omega_i [D_i^{\mathrm{T}} (\nabla^{\perp} T \times \hat{\boldsymbol{B}}) + D_i^{\mathrm{TH}} \nabla^{\perp} T] \right\} \\
&- \sum_{j \neq i} \frac{n_i n_j}{n^2 T \mathcal{D}_{ij}} \left\{ \frac{1}{n_j m_j} [D_j^{\mathrm{T}\perp} \nabla^{\perp} T + D_j^{\mathrm{TH}} (\boldsymbol{B} \times \nabla^{\perp} T)] - \frac{1}{n_i m_i} [D_i^{\mathrm{T}\perp} \nabla^{\perp} T + D_i^{\mathrm{TH}} (\hat{\boldsymbol{B}} \times \nabla^{\perp} T)] \right\}
\end{aligned} \tag{43}$$

where the first two terms involving $D_i^{\mathrm{T}\perp}$ and D_i^{TH} are, respectively, the thermal diffusion contributions to $\boldsymbol{j} \times \boldsymbol{B}$ and $\langle \boldsymbol{V}_i \rangle \times \boldsymbol{B}$. It is clear that Eqs. (34) and (43) could be added together to obtain equations for the total diffusion velocities, if desired.

Equation (43), because of its simpler form, is much more readily applicable to many problems than Eq. (31). However, caution must be exercised, since the first approximation to D_{ij} has been used in its derivation. In some cases, the best known being the electron diffusion in a fully ionized gas [3, 19] this approximation is highly inaccurate. At low ionization in gases where the electron-atom cross section displays the Ramsauer effect, a much higher approximation to the electron diffusion coefficients is needed [17, 40]. Fortunately, it is possible to derive a much simplified form of Eq. (31) and the corresponding transport coefficients for electrons, which can be used to determine the electron diffusion velocity. Equation (43) can still be used for the diffusion velocities of the ions and neutral species. In the case of the diffusion of one type of ion through other ions, errors of about 18% in the ordinary diffusion flux may be introduced by using Eq. (43). Differences of this size were found between the first and higher approximations to the self-diffusion coefficients of gas composed only of ions [19]. In the case of ions diffusing through atoms, or excited atoms diffusing through groundstate atoms, this equation should be accurate within a few percent.

[40] R. S. Devoto, Phys. Fluids 10, 354 [1967]; 8th Intern. Conf. on Phenomena in Ionized Gases, Vienna 1967.

Heat Flux

The heat flux is given by

$$\begin{aligned}
q &\equiv \tfrac{1}{2} \sum_j m_j \int \mathbf{V}_j V_j^2 f_j^{(0)} \varphi_j \, \mathrm{d}\mathbf{V}_j \\
&= \tfrac{5}{2} k\,T \sum_j n_j \langle \mathbf{V}_j \rangle - \lambda'^{\parallel} \nabla^{\parallel} T - \lambda'^{\perp} \nabla^{\perp} T \\
&\qquad - \lambda'^{\mathrm{H}} (\hat{\boldsymbol{B}} \times \nabla^{\perp} T) \\
&- n\,k\,T \sum_j \frac{1}{n_j\, m_j} [D_j^{\mathrm{T}\parallel} \mathbf{d}_j^{\parallel} + D_j^{\mathrm{T}\perp} \mathbf{d}_j^{\perp} \\
&\qquad + D_j^{\mathrm{TH}} (\hat{\boldsymbol{B}} \times \mathbf{d}_j^{\perp})].
\end{aligned} \tag{44}$$

The derivation of this equation for the parallel components has been already given in Ref. [21]. The derivation for the other components follows in much the same way. In the reduction of the last terms in Eq. (47) it is convenient to work with the complex quantity $D_j^{T\perp} + i D_j^{\mathrm{TH}}$, rather than with each component separately, so that the complex Eq. (15) and its equivalent for A_i may be directly used. It is also necessary to employ the auxiliary conditions on $C_i^h - C_i^k$ and A_i to effect the reduction.

The parallel coefficient $\lambda'^{\parallel}$ is given by the expressions for the un-ionized gas, while the perpendicular and Hall coefficents are given by

$$\lambda' = \lambda'^{\perp} + i\,\lambda'^{\mathrm{H}} = -\tfrac{5}{4}\,k(2\,k\,T)^{1/2} \sum_j \frac{n_j\,a_{j1}}{m_j^{1/2}} \tag{45}$$

and thus can be found from the usual expression [41] upon replacement of the real elements by those of Eq. (28). These coefficients are not the true thermal conductivities, since these are conventionally defined as the coefficient of the thermal gradient when all diffusion fluxes vanish. The various components of $\boldsymbol{d}_j$ must therefore be eliminated in favour of the $\langle \boldsymbol{V}_i \rangle$ and ∇T with the aid of Eq. (31). MUCKENFUSS and CURTISS [42] accomplished this for the second approximation, i. e. for two terms in the Sonine polynomial expansion [see Eq. (16)]; it was recently found possible to extend their proof to an arbitrary level of approximation [43]. Through the use of complex arithmetic, these proofs may be easily extended to the gas in a magnetic field. Only the simpler proof for the second approximation will be considered here.

In place of the real vector $\boldsymbol{\alpha}_i$ of Muckenfuss and Curtiss we define the complex vector $\boldsymbol{\alpha}_i^*$ by

$$\begin{aligned}\boldsymbol{\alpha}_1^* &= \frac{2}{n}\left(\frac{m_i}{2\,k\,T}\right)^{1/2}\left[\langle \boldsymbol{V}_i^* \rangle + \frac{1}{n_i\,m_i\,T}\,D_i^{\mathrm{T}}\,\nabla^*\,T\right] \\ &= \sum_j c_{io}^{jh}\,\boldsymbol{d}_j^* \end{aligned} \tag{46}$$

where D_i^{T} and c_{io}^{jh} are, of course, also complex. We may now solve for $\boldsymbol{d}_i^*$ as done by Muckenfuss and sion for $\boldsymbol{q}^{\perp} - i\,\hat{\boldsymbol{B}} \times \boldsymbol{q}^{\perp}$ which may be formed from Curtiss, and substitute the solution into an expres- Eq. (44). Extracting the real part of this expression, we obtain finally for the heat flux the expression

$$\begin{aligned}\boldsymbol{q} = \sum_j \Bigg\{ &\left[\frac{5}{2}\,k\,T - \frac{\varrho\,k\,T}{n}\sum_i \frac{D_i^{\mathrm{T}\parallel}\,E_{ij}^{\parallel}}{n_i\,m_i}\right] n_j \langle \boldsymbol{V}_j^{\parallel} \rangle \\ + &\left[\frac{5}{2}\,k\,T - \frac{\varrho\,k\,T}{n}\sum_i \frac{D_i^{\mathrm{T}\perp}\,E_{ij}^{\perp} - D_i^{\mathrm{TH}}\,E_{ij}^{\mathrm{H}}}{n_i\,m_i}\right] n_j \langle \boldsymbol{V}_j^{\perp} \rangle \\ &- \frac{\varrho\,k\,T}{n}\sum_i \frac{D_i^{\mathrm{TH}}\,E_{ij}^{\perp} + D_i^{\mathrm{T}\perp}\,E_{ij}^{\mathrm{H}}}{n_i\,m_i}\, n_j (\hat{\boldsymbol{B}} \times \langle \boldsymbol{V}_j^{\perp} \rangle) \Bigg\} \\ &- \lambda^{\parallel}\,\nabla^{\parallel}\,T - \lambda^{\perp}\,\nabla^{\perp}\,T - \lambda^{\mathrm{H}}(\hat{\boldsymbol{B}} \times \nabla^{\perp}\,T).\end{aligned} \tag{47}$$

The complex thermal conductivity $\lambda^{\perp} + i\,\lambda^{\mathrm{H}}$ is given by the expression for $\lambda^{\parallel}$ with replacement of determinant real elements by those given in Eq. (28) [44]. The matrix E_{ij} is the inverse of the (complex) matrix $D_{ij}\,m_j$ [41]

$$E_{ij} \equiv E_{ij}^{\perp} + i\,E_{ij}^{\mathrm{H}} = \frac{\operatorname{adj}(D_{ij}\,m_j)}{\det(D_{ij}\,m_j)}\,. \tag{48}$$

$E_{ij}^{\parallel}$ is the limit of $E_{ij}^{\perp}$ as $B \to 0$, or may be computed directly from the real diffusion coefficients. Unfortunately, no simple relations corresponding to those for $\lambda^{\parallel}$, $\lambda^{\perp}$ and λ^{H} have been found for the terms multiplying the diffusion velocities in the general multicomponent mixture. Simple expressions are available for the binary gas [42] and for the contribution of the electrons to this term [17, 45].

Comparison with Species Momentum Equations

In place of the diffusion velocity of each species $\langle \boldsymbol{V}_i \rangle$ SCHLÜTER [25] considers the average velocity

$$\langle \boldsymbol{v}_i \rangle = \frac{1}{n_i}\int \boldsymbol{v}_i\,f_i\,\mathrm{d}\boldsymbol{v}_i = \langle \boldsymbol{V}_i \rangle - \boldsymbol{v}_0\,.$$

An equation for $\langle \boldsymbol{v}_i \rangle$ may be developed from the equation of change of a molecular property [1],

$$\begin{aligned} m_i\,n_i \frac{D_i \langle \boldsymbol{v}_i \rangle}{D_t} &= -\nabla\,p_i' - \nabla \cdot \boldsymbol{\pi}_i' + n_i\,\boldsymbol{X}_i \\ &\quad + n_i\,e_i\left(\boldsymbol{E} + \frac{\langle \boldsymbol{v}_i \rangle \times \boldsymbol{B}}{c}\right) + \boldsymbol{R}_i \end{aligned} \tag{47}$$

with

$$p_i' = \tfrac{1}{3}\,n_i\,m_i\,\langle\,(\boldsymbol{v}_i - \langle \boldsymbol{v}_i \rangle)^2\,\rangle\,, \tag{48}$$

$$\pi_{i\alpha}' = n_i\,m_i \langle\,(v_{i\alpha} - \langle v_{i\alpha} \rangle)\;(v_{i\beta} - \langle v_{i\beta} \rangle)\,\rangle - p_i'\,\delta_{\alpha\beta}\,, \tag{49}$$

$$\boldsymbol{R}_i = m_i \sum_j \iiint (\boldsymbol{v}_i - \langle \boldsymbol{v}_i \rangle)\,f_i\,f_j\,g\,\sigma\,\mathrm{d}\Omega\,\mathrm{d}\boldsymbol{v}_j\,\mathrm{d}\boldsymbol{v}_i \tag{50}$$

and

$$\frac{D_i}{D_t} = \frac{\partial}{\partial t} + \langle \boldsymbol{v}_i \rangle \cdot \nabla. \tag{51}$$

Before application of Eq. (48) to a specific problem, two assumptions are usually made [25, 27, 28]:

$$\pi_{i\alpha\beta}' = 0\,, \tag{52 a}$$

$$\boldsymbol{R}_i = -n_i \sum_j n_j\,\varepsilon_{ij}(\langle \boldsymbol{v}_i \rangle - \langle \boldsymbol{v}_j \rangle). \tag{52 b}$$

[41] See Eq. (14) of Ref. [19] or the equivalent equations in Ref. [21].
[42] C. MUCKENFUSS and C. F. CURTISS, J. Chem. Phys. **29**, 1273 [1958].
[43] R. S. DEVOTO, to be published.
[44] See Eq. (30) of Ref. [42], or the equivalent in Ref. [21] or [43].
[45] R. S. DEVOTO, paper to follow.

In (52 a) the stress tensor is neglected in comparison with the static pressure. The form assumed in (52 b) for $\boldsymbol{R}_i$ includes only a friction between the species, with the constants ε_{ij} to be specified from kinetic theory. A thermal force was also introduced by MAECKER and PETERS [46] in an attempt to allow for thermal diffusion in the mixture. It can be easily shown from the expressions of the previous sections or from work by BRAGINSKII [6, 33] that the form which they use cannot describe thermal diffusion perpendicular to a magnetic field. The thermal diffusion will therefore be neglected in this section.

With the above assumptions Eq. (48) becomes

$$n_i\, m_i \frac{\mathrm{D}_i \langle \boldsymbol{v}_i \rangle}{\mathrm{D}t} = -\nabla p_i' + n_i\, \boldsymbol{X}_i + n_i\, e_i \left(\boldsymbol{E} + \frac{\langle \boldsymbol{v}_i \rangle \times \boldsymbol{B}}{c} \right) - n_i \sum_j n_j\, \varepsilon_{ij} (\langle \boldsymbol{v}_i \rangle - \langle \boldsymbol{v}_j \rangle). \tag{53}$$

The presence of the velocity difference in Eq. (53) suggests that we compare it with Eqs. (34) and (43). They differ from Eq. (53) in several respects, the most noticeable being the lack of a substantial differential. We can introduce such a substantial derivative with an equation used in the development of the left-hand side of Eq. (1) [47]

$$\varrho \frac{\mathrm{D}\boldsymbol{v}_0}{\mathrm{D}t} \equiv \varrho \left(\frac{\partial \boldsymbol{v}_0}{\partial t} + \boldsymbol{v}_0 \cdot \nabla\, \boldsymbol{v}_0 \right) = -\nabla p + \frac{\boldsymbol{j} \times \boldsymbol{B}}{c} + \sum_k n_k\, \boldsymbol{X}_k\,. \tag{54}$$

This equation is, of course, the momentum equation for a gas in the absence of a stress tensor, in the present case with a Maxwellian distribution of velocities. Since the forces causing the diffusion come from the left-hand side of Eq. (1), we may reverse this step and use Eq. (54) to rewrite Eq. (3) for $\boldsymbol{d}_i$. Combining Eqs. (3), (34), (43) and (54) under the assumption of charge neutrality ($\varrho_c = 0$), we obtain

$$n_i\, m_i \frac{\mathrm{D}\boldsymbol{v}_0}{\mathrm{D}t} = -\nabla p_i + n_i\, \boldsymbol{X}_i + n_i\, e_i \left(\boldsymbol{E} + \frac{\langle \boldsymbol{v}_i \rangle \times \boldsymbol{B}}{c} \right) - \sum_j \frac{n_i\, n_j\, k\, T}{n\, D_{ij}} (\langle \boldsymbol{v}_i \rangle - \langle \boldsymbol{v}_j \rangle). \tag{55}$$

From a comparison of Eq. (53) and (55) we can choose ε_{ij} as

$$\varepsilon_{ij} = k\, T / n\, D_{ij} \tag{56}$$

in agreement with other workers. However, there are two discrepancies between Eqs. (53) and (55), the most evident being that between the substantial derivatives. They differ by the amount

$$\frac{\partial \langle V_i \rangle}{\partial t} + \boldsymbol{v}_0 \cdot \nabla \langle \boldsymbol{V}_i \rangle + \langle \boldsymbol{V}_i \rangle \cdot \nabla (\boldsymbol{v}_0 + \langle \boldsymbol{V}_i \rangle). \tag{57}$$

When $\boldsymbol{v}_0$ vanishes, or is small, and forces causing diffusion are present and possibly changing with time, then it is conceivable that Eq. (57) would not vanish. Fortunately, in many problems treated with Eq. (53), only the steady state is considered, and the other terms vanish identically or are indeed negligible.

The second discrepancy occurs in definition of partial pressure. From the relation between the various velocities we can write

$$p_i' = p_i - \tfrac{1}{3}\, n_i\, m_i \langle V_i \rangle^2 \tag{48 a}$$

where

$$p_i = \tfrac{1}{3}\, n_i\, m_i \langle V_i^2 \rangle$$

is the usual definition of partial pressure [1]. The difference can be seen to arise from different definitions of the "thermal" velocity of the gas: in the primed system it is defined relative to the species average velocity $\langle \boldsymbol{v}_i \rangle$, in the unprimed relative to the mass-average velocity $\boldsymbol{v}_0$. If we sum Eq. (48 a), we see that the total pressures also differ

$$p' = p - \tfrac{1}{3} \cdot \sum_i n_i\, m_i \langle V_i \rangle^2.$$

Although it is possible to think of situations in which the different definitions could be important (e. g. $\nabla p_i = 0$ but $\nabla p_i' \neq 0$), in general they will be negligible. In the first order Chapman-Enskog theory it is assumed that the various gradients and forces are small enough to neglect quadratic terms compared to the linear terms. The diffusion velocities are proportional to the gradients, so terms quadratic in those velocities should indeed be negligible. It may also be noted here that the stress tensors and the temperatures in the two systems also differ by terms proportional to the square of the diffusion velocities.

The derivation of Eq. (55) has also been carried out in a slightly different manner by JOHNSON [26]; however, he did not consider the species momentum equation and thus neglected the important differences brought out here. His derivations, as well as the

[46] H. MAECKER and TH. PETERS, Z. Phys. **144**, 586 [1956].

[47] See Ref. [1], pp. 330–331 for a discussion of this step.

present one, is based on the use of the first approximation, which restricts the application of this equation to species where this level of approximation is accurate. We should not expect to obtain numerically correct factors if we use the above to compute the electron flux in a gas. BRAGINSKII [6, 33], however, was able to derive an equation similar to (47) for the electron in a fully-ionized gas with inclusion of thermal diffusion forces in $\boldsymbol{R}_i$. The above derivation of Eq. (55) from kinetic theory allows us to set quite generally its limits of validity, namely, those of the Chapman-Enskog method: mean-free-path and time between collisions shorter than distances and times characteristic of the fluid gradients, and those mentioned earlier with regard to the collision integral and thermal diffusion. Such general statements are not possible in the derivation from a species momentum equation. The species momentum equation does have the advantage of being able to describe wave phenomenon in which electron or ion inertia is important. Such cannot be accomplished with the usual Chapman-Enskog method.

It is evident from Eqs. (34) and (43) that we could have included terms representing thermal diffusion in Eq. (55) [48]. Such a step does not seem advantageous for several reasons. Firstly such terms in these equations perpendicular to the magnetic field are quite complicated, and would be difficult to use in practical applications. Secondly, since there is no reasonably simple relation between the thermal diffusion and binary diffusion coefficients, the full multicomponent expressions described earlier must be used to compute D_i^{T}. But then much of the simplicity of the method of this section is lost and one would be better-advised to use instead Eq. (31) for the diffusion velocity. This becomes more evident when we consider that D_i^{T} will almost always be computed with a higher approximation than D_{ij} so little additional effort is required to compute the multicomponent diffusion coefficients simultaneously with the computation of those for thermal diffusion.

[48] For an example where thermal diffusion must be considered see O. KLÜBER, Z. Naturforsch. **22 a**, 1599 [1967], P. H. GRASSMANN, Z. Naturforsch. **23 a**, 251 [1968] and J. RAEDER and S. WIRTZ, Z. Naturforsch. **23 a**, 1695 [1968].

Discussion

We have seen how it is possible to modify the usual expressions for transport coefficients of multicomponent gases to alow for ionization in the presence of a magnetic field. The modifications involve only the introduction of complex elements containing the cyclotron frequencies into the determinants. Nonetheless, the practical application of these expressions to a plasma containing electrons becomes a complicated task since, in order to compute accurate coefficients for the mixture, at least the second approximation must be used for the electrical conductivity and the third approximation for the thermal conductivity of ionized gases. This slow convergence, relative to un-ionized gases, is caused by the presence of the electrons in the mixture. It is possible to separate the contribution of the electrons to the transport coefficients from that of the ions and neutrals, and so to use a lower approximation for the latter than for the electrons. The present expressions are then valuable for computing the properties of the mixture of ions and neutrals, and as a starting point for deducing simplified expressions for the electron properties. These simplified expressions will be developed in a paper to follow.

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